GENERAL

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Applied Analysis by the Hilbert Space Method: An Introduction with Application to the Wave, Heat, and Schrödinger Equations in Pure and Applied Mathematics

Modern mathematics has long recognized that the language of functional analysis—particularly Hilbert space theory—provides a unifying framework for studying partial differential equations (PDEs). The Hilbert space method, rooted in the geometry of inner products and orthogonality, allows mathematicians to transform seemingly intractable differential problems into operator equations amenable to spectral analysis, semigroup theory, and variational techniques. This article offers a comprehensive, yet accessible, exploration of the Hilbert space approach to PDEs, with a focus on three canonical equations: the wave equation, the heat equation, and the Schrödinger equation. We will discuss the theoretical underpinnings, present key theorems, illustrate practical applications, and highlight how these ideas bridge pure and applied mathematics.

Why Hilbert Spaces Matter in Applied Analysis

Hilbert spaces are complete inner-product spaces that generalize Euclidean geometry to infinite dimensions. Their completeness ensures convergence of Cauchy sequences, while the inner product induces notions of length and angle, enabling orthogonal decompositions. In applied analysis, these properties allow us to:

- Define and study linear operators that model physical systems.
- Use spectral theory to analyze eigenvalues and eigenfunctions, which often represent natural modes of vibration or diffusion.
- Employ semigroup theory to describe time evolution in evolution equations.
- Apply variational principles to derive weak formulations suitable for numerical approximation.

Because PDEs frequently arise in physics, engineering, and finance, the Hilbert space method becomes an indispensable tool for both theoretical insight and computational implementation.

Foundations of Hilbert Space Analysis

A Hilbert space \\(H\\) is a vector space equipped with an inner product \\(\\langle \\cdot,\\cdot \\rangle\\) such that every Cauchy sequence converges to an element in \\(H\\). Key properties include:

- Orthogonality: Two vectors \\(u, v\\) are orthogonal if \\(\\langle u, v \\rangle = 0\\).
- Projection Theorem: For any closed subspace \\(M \\subset H\\), each \\(f \\in H\\) can be uniquely decomposed as \\(f = m + n\\) with \\(m \\in M\\) and \\(n \\in M^\\perp\\).
- Riesz Representation: Every continuous linear functional on \\(H\\) can be expressed as \\(\\langle \\cdot, g \\rangle\\) for a unique \\(g \\in H\\).

In Hilbert space analysis, we often study densely defined linear operators \\(A: D(A) \\subset H \\to H\\). Important concepts include:

- Self-Adjointness: \\(A = A^*\\) ensures real spectrum and orthogonal eigenvectors.

- Compactness: Compact operators have discrete spectra accumulating only at zero.
- Resolvent Set: \\(\\rho(A) = \\{ \\lambda \\in \\mathbb{C} : (A - \\lambda I)^{-1}\\ \\text{exists and is bounded}\\}\\).
- Spectral Theorem: For self-adjoint \\(A\\), there exists a projection-valued measure \\(E\\) such that \\(A = \\int \\lambda \\, dE(\\lambda)\\).

From Differential Equations to Operator Equations

Consider a linear PDE such as the wave equation \\(u_{tt} = c^2 \\Delta u\\). By introducing the state vector \\(U = (u, u_t)\\), we can rewrite the PDE as an abstract Cauchy problem:

\\[\n\\frac{d}{dt} U(t) = \\mathcal{A} U(t), \\quad U(0) = U_0,\n\\]

where \\(\\mathcal{A}\\) is an unbounded operator defined on a suitable Hilbert space. The solution can then be expressed via the exponential of \\(\\mathcal{A}\\) if \\(\\mathcal{A}\\) generates a \\(C_0\\)-semigroup. Similar transformations apply to the heat and Schrödinger equations.

Application to the Heat Equation

Let \\(\\Omega \\subset \\mathbb{R}^n\\) be a bounded domain with smooth boundary. The heat equation reads:

\\[\n\\begin{cases}\nu_t(t,x) - \\Delta u(t,x) = f(t,x), & (t,x) \\in (0,T] \\times \\Omega, \\\\\nu(0,x) = u_0(x), & x \\in \\Omega, \\\\\nu(t,x) = 0, & (t,x) \\in (0,T] \\times \\partial\\Omega.\n\\end{cases}\n\\]

Define \\(H = L^2(\\Omega)\\) and \\(V = H_0^1(\\Omega)\\). The bilinear form \\(a(u,v) = \\int_\\Omega \\nabla u \\cdot \\nabla v\\,dx\\) is coercive on \\(V\\). The weak solution satisfies:

\\[\n(u_t(t), v) + a(u(t), v) = (f(t), v) \\quad \\forall v \\in V.\n\\]

Using the Riesz representation, we identify the operator \\(A: V \\to V^*\\) via \\(\\langle Au, v \\rangle = a(u,v)\\). Since \\(A\\) is self-adjoint and positive definite, the abstract evolution equation \\(u'(t) + Au(t) = f(t)\\) has a unique solution in \\(C([0,T];H) \\cap L^2(0,T;V)\\).

Let \\(A\\) be the Dirichlet Laplacian. The heat semigroup \\(e^{-tA}\\) is a contraction semigroup on \\(H\\). The mild solution is:

\\[\nu(t) = e^{-tA}u_0 + \\int_0^t e^{-(t-s)A} f(s)\\,ds.\n\\]

This representation is invaluable for numerical schemes such as exponential integrators and for proving regularity results.

Application to the Wave Equation

The classical wave equation is:

\\[\n\\begin{cases}\nu_{tt}(t,x) - c^2 \\Delta u(t,x) = 0, & (t,x) \\in (0,T] \\times \\Omega, \\\\\nu(0,x) = u_0(x), \\quad u_t(0,x) = v_0(x), & x \\in \\Omega, \\\\\nu(t,x) = 0, & (t,x) \\in (0,T] \\times \\partial\\Omega.\n\\end{cases}\n\\]

Introduce \\(U = (u, u_t)\\) and define the operator matrix:

\\[\n\\mathcal{A} = \\begin{pmatrix}\n0 & I \\\\\nc^2 \\Delta & 0\n\\end{pmatrix},\n\\quad D(\\mathcal{A}) = \\{ (u,v) \\in H_0^1(\\Omega) \\times H^1_0(\\Omega) \\}.\n\\]

The abstract Cauchy problem \\(U'(t) = \\mathcal{A} U(t)\\) with \\(U(0) = (u_0, v_0)\\) generates a unitary group on \\(H = H_0^1(\\Omega) \\times L^2(\\Omega)\\). The solution can be written as \\(U(t) = e^{t\\mathcal{A}} U_0\\).

The energy functional \\(E(t) = \\frac{1}{2}\\|u_t(t)\\|_{L^2}^2 + \\frac{c^2}{2}\\|\\nabla u(t)\\|_{L^2}^2\\) remains

constant over time, reflecting the unitary nature of the evolution. Spectral decomposition of the Laplacian yields eigenpairs \\((\\lambda_k, \\phi_k)\\). The solution expands as:

\\[\nu(t,x) = \\sum_{k=1}^\\infty \\left(a_k \\cos(c\\sqrt{\\lambda_k}t) + b_k \\sin(c\\sqrt{\\lambda_k}t) \\right) \ \phi_k(x),\n\\]

where \\(a_k = (u_0,\\phi_k)\\) and \\(b_k = (v_0,\\phi_k)/ (c\\sqrt{\\lambda_k})\\). This modal analysis is fundamental in structural engineering and acoustics.

Application to the Schrödinger Equation

The linear Schrödinger equation is:

\\[\n\\begin{cases}\ni \\partial_t \\psi(t,x) = -\\frac{\\hbar^2}{2m} \\Delta \\psi(t,x) + V(x)\\psi(t,x), & (t,x) \\in (0,T] \ \times \\mathbb{R}^n, \\\\\n\\psi(0,x) = \\psi_0(x).\n\\end{cases}\n\\]

We assume \\(V \\in L^\\infty_{\\text{loc}}\\) and \\(\\psi_0 \\in L^2(\\mathbb{R}^n)\\).

Define the Hamiltonian operator \\(H = -\\frac{\\hbar^2}{2m}\\Delta + V\\) with domain \\(D(H) = H^2(\\mathbb{R}^n)\ \). Under standard assumptions on \\(V\\), \\(H\\) is self-adjoint on \\(L^2(\\mathbb{R}^n)\\). The Schrödinger equation becomes:

\\[\ni \\psi'(t) = H \\psi(t).\n\\]

By Stone's theorem, the one-parameter family \\(U(t) = e^{-itH/\\hbar}\\) forms a strongly continuous unitary group on \\(L^2(\\mathbb{R}^n)\\). The solution is given by the propagator:

\\[\n\\psi(t) = U(t)\\psi_0 = e^{-itH/\\hbar}\\psi_0.\n\\]

In free space (\\(V=0\\)), the propagator reduces to convolution with the fundamental solution, leading to explicit Fourier integral representations.

The spectrum of \\(H\\) determines bound states (discrete spectrum) and scattering states (continuous spectrum). The spectral theorem yields the spectral decomposition:

\\[\n\\psi(t) = \\int_{\\sigma(H)} e^{-it\\lambda/\\hbar} dE_\\lambda \\psi_0,\n\\]

where \\(E_\\lambda\\) is the projection-valued measure associated with \\(H\\). This framework underpins quantum scattering theory and the study of resonances.

Bridging Pure and Applied Mathematics

Hilbert space methods provide a common language that unites rigorous theoretical analysis with practical computational techniques. For instance:

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